TOPOLGY AND GEOMETRY (SAMPLE ASSIGNMENT)

1. Let (M, ω) be a symplectic manifold, J a compatible almost-complex structure, and g the corresponding Riemannian metric. Show that two-dimensional almost-complex submanifolds of M are absolutely volume minimizing in their homology class, i.e.: let C, C' be two-dimensional compact closed oriented submanifolds of M, representing the same homology class $[C] = [C'] \in H_2(M, \mathbb{Z})$. Assume that $J(TC) = TC$ (and the orientation of C agrees with that induced by J). Then $\mathrm{vol}_g(C) \leq \mathrm{vol}_g(C')$.

1. Given a point $p \in C'$ (a two-dimensional oriented submanifold), let (e, f) be an oriented basis of T_pC', orthonormal with respect to the metric g induced by ω and J. Then $\omega(e, f) = g(Je, f) \leq |Je| \ f| = |e \ \ f| = 1$. Meanwhile, the area form $dvol_{g\ C'}$ induced by g on C' is given by $dvol_{g|C'}(e, f) = 1$. Hence $\omega_{|C'} \leq dvol_{g\ C'}$ at every point of C'; integrating, we deduce that $[\omega] \cdot [C'] = \int_{C'} \omega \leq vol_g(C')$.

In the case of C (an almost-complex submanifold, equipped with the orientation induced by J), an oriented orthonormal basis of T_pC is given by (e, Je) where e is any unit length vector in T_pC. (Note that $|Je| = |e| = 1$ and $g(Je, e) = \omega(e, e) = 0$). Then $\omega(e, Je) = g(Je, Je) = 1 = dvol_{g|C}(e, Je)$, so $\omega_{|C} = dvol_{g|C}$, and $[\omega] \cdot [C] = \int_C \omega = vol_g(C)$.

In conclusion, $vol_g(C) = [\omega] \cdot [C] = [\omega] \cdot [C'] \leq vol_g(C')$.

2. We will admit the fact that the cohomology ring of $\mathbb{CP}^n$ (the set of complex lines through 0 in $\mathbb{C}^{n+1}$) is $H^*(\mathbb{CP}^n, \mathbb{Z}) = \mathbb{Z}[h]/h^{n+1}$, where $h \in H^2(\mathbb{CP}^n, \mathbb{Z})$ is Poincaré dual to the homology class represented by a linear $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$.

The tautological line bundle $L \to \mathbb{CP}^n$ is the subbundle of the trivial bundle $\mathbb{C}^{n+1} \times \mathbb{CP}^n$ whose fiber at a point of $\mathbb{CP}^n$ is the corresponding line in $\mathbb{C}^{n+1}$. The homogeneous coordinates on $\mathbb{CP}^n$ are actually sections of the dual bundle L^*. (Convince yourself of this).

a) Show that $c_1(L) = -h$, and show that the direct sum of $T\mathbb{CP}^n$ with the trivial line bundle $\mathbb{C}$ is isomorphic to the direct sum of $n+1$ copies of L^*. From this, deduce the Chern classes of the tangent bundle $T\mathbb{CP}^n$.

b) Let $X \subset \mathbb{CP}^n$ be a smooth complex hypersurface of degree d, i.e. the submanifold defined by the equation $P(z_0, \ldots, z_n) = 0$ where P is a homogeneous polynomial of degree d (transverse to the zero section, i.e. with nonvanishing differential along its zero set). Show that $T\mathbb{CP}^n{}_{|X} = TX \oplus (L^*)^{\otimes d}_{|X}$, and deduce the Chern classes of TX.

2. a) The homogeneous coordinate x_n is a linear form on $\mathbb{C}^{n+1}$ (namely, $(x_0, \ldots, x_n) \mapsto x_n$) and hence, by restriction to the tautological line, a linear form on L. This section of L^* vanishes precisely at those points $[x_0 : \cdots : x_n]$ for which the last coordinate is zero, so its zero set is $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$. Moreover, it vanishes transversely, and the orientation induced on its zero set is the natural one (because all orientations agree with those induced by the complex structure). So $c_1(L^*) = e(L^*)$ is Poincaré dual to $[\mathbb{CP}^{n \ \ 1}] \in H_{2n-2}(\mathbb{CP}^n)$, i.e. $c_1(L^*) = h$. Therefore $c_1(L) = -c_1(L^*) = -h$.

Given a line $\ell \subset \mathbb{C}^{n+1}$ (defining a point $p = [\ell] \in \mathbb{CP}^n$), any nearby line can be parametrized by a map $\ell \to \mathbb{C}^{n+1}$, $x \mapsto x + u(x)$, where $u \in \mathrm{Hom}(\ell, \mathbb{C}^{n+1})$. This gives a map (in fact a local submersion) $\psi : \mathrm{Hom}(\ell, \mathbb{C}^{n+1}) \to \mathbb{CP}^n$ defined by $\psi(u) = [\mathrm{Im}(\mathrm{Id} + u)]$. Its differential at the origin is $\Psi = d_0\psi : \mathrm{Hom}(\ell, \mathbb{C}^{n+1}) \to T_p\mathbb{CP}^n$. We claim that Ψ is surjective, with kernel $\mathrm{Hom}(\ell, \ell) \simeq \mathbb{C}$ (those linear maps whose image is contained in ℓ). Indeed, this can be checked easily in the case where ℓ is the first coordinate axis, and $\psi((u_0, \ldots, u_n)) = [1 + u_0 : u_1 : \cdots : u_n]$. Therefore, we have a short exact sequence of vector bundles $0 \to \mathbb{C} = \mathrm{Hom}(L, L) \to \mathrm{Hom}(L, \mathbb{C}^{n+1}) = (L^*)^{n+1} \to T\mathbb{CP}^n \to 0$ (where $\mathbb{C}$ denotes h l l b dl $\mathbb{CP}^n$)

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Taking a complement F to the subbundle $\mathrm{Hom}(L,L) \subset \mathrm{Hom}(L,\mathbb{C}^{n+1})$ (e.g. its orthogonal complement for some Hermitian metric), the restriction of Ψ to F is an isomorphism, so we conclude that $\mathrm{Hom}(L,\mathbb{C}^{n+1}) = (L^*)^{n+1}$ is isomorphic to $T\mathbb{CP}^n \oplus \mathbb{C}$.

Since Chern classes behave multiplicatively under direct sums, and $c(L^*) = 1 + c_1(L^*) = 1 + h$, we have $c(T\mathbb{CP}^n) = c(T\mathbb{CP}^n \oplus \mathbb{C}) = c((L^*)^{n+1}) = (1+h)^{n+1}$. Expanding into powers of h, we deduce that $c_k(T\mathbb{CP}^n) = \binom{n+1}{k} h^k$ for all $1 \le k \le n$.

b) Consider $X = P^{-1}(0)$, where P is a homogeneous polynomial of degree d in the homogeneous coordinates, i.e. a section of $(L^*)^{\otimes d}$. Fix any connection on $(L^*)^{\otimes d}$. If we assume that P is transverse to the zero section, then at any point $x \in X$ the linear map $(\nabla P)_x : T_x\mathbb{CP}^n \to (L^*)^{\otimes d}_x$ (which does not depend on the chosen connection since $P(x) = 0$) is surjective and its kernel is T_xX (see Homework 2). Therefore we get a short exact sequence of vector bundles $0 \to TX \to T\mathbb{CP}^n_{|X} \to (L^*)^{\otimes d}_{|X} \to 0$, and considering again a complement to TX in $T\mathbb{CP}^n_{X}$ we conclude that $T\mathbb{CP}^n_{X} \simeq TX \oplus (L^*)^{\otimes d}_{|X}$.

Using additivity of the first Chern class of a line bundle under tensor product, we have $c_1((L^*)^{\otimes d}) = 1 + dh$. Let $\alpha = h\ _X \in H^2(X,\mathbb{Z})$ (the pullback of h by the inclusion $i : X \hookrightarrow \mathbb{CP}^n$). Using the multiplicativity of Chern classes under direct sums and their functoriality under pullback, we deduce that $(1+\alpha)^{n+1} = c(TX)\cdot(1+d\alpha)$.

Since $\alpha^n = 0$ in the cohomology of X (for dimension reasons), $1 + d\alpha$ is invertible, with inverse $(1+d\alpha)^{-1} = \sum_{k=0}^{n-1}(-1)^k d^k\,\alpha^k$. The total Chern class of TX is then $1 + c_1(TX) + \cdots + c_{n-1}(TX) = (1+d\alpha)\ ^1(1+\alpha)^{n+1}$.

3. Let M be a compact oriented 4-manifold, equipped with a Riemannian metric g. A 2-form is said to be *selfdual* if $*\alpha = \alpha$, *antiselfdual* if $*\alpha = \quad \alpha$. The bundles of selfdual (resp. antiselfdual) 2-forms are denoted by $\Lambda^2_+T^*M$ and $\Lambda^2\ T^*M$ respectively.

a) Show that the Hodge $*$ operator induces a decomposition of the space of harmonic forms $\mathcal{H}^2 = \mathcal{H}^2_+ \oplus \mathcal{H}^2_-$ into selfdual and antiselfdual harmonic forms. Show that, with respect to the intersection pairing $(\alpha,\beta) \mapsto \int_M \alpha\wedge\beta$, these summands are definite positive (resp. definite negative) and orthogonal to each other.

b) Assume that (M,ω) is a compact Kähler manifold of real dimension 4. Show that $\Lambda^2_+T^*M \otimes \mathbb{C} = \Lambda^{2,0} \oplus \Lambda^{0,2} \oplus \mathbb{C}\omega$, where the summands are orthogonal to each other, and $\Lambda^2_-T^*M \otimes \mathbb{C} = \omega^\perp \subset \Lambda^{1,1}$. Deduce that the space of real harmonic (1,1)-forms is $\mathcal{H}^{1,1}_{\mathbb{R}} = \mathcal{H}^2_- \oplus \mathbb{R}\omega$.

(Since algebraic curves in a complex projective surface are Poincaré dual to classes in $NS := H^{1,1}(M) \cap H^2(M,\mathbb{Z})$, this implies the *Hodge index theorem*, which asserts that the intersection pairing on algebraic cycles in a complex projective surface has signature $(1, \dim NS - 1)$).

3. a) The Hodge $*$ operator on $\Omega^2(M^4)$ satisfies $*^2 = 1$, and every 2-form α decomposes into the sum of a selfdual part $\alpha^+ = \frac{1}{2}(\alpha + *\alpha)$ and an antiselfdual part $\alpha^- = \frac{1}{2}(\alpha - *\alpha)$. On an even-dimensional manifold, $d^* = -*d*$ in all degrees, so $\Delta = dd^* + d^*d = -d*d* - *d*d$ commutes with $*$. Therefore, if α is harmonic then so is $*\alpha$, and hence so are α^+ and α^-.

So every harmonic form α decomposes into the sum of a harmonic selfdual form (α^+) and a harmonic antiselfdual form ($\alpha\ $). Moreover, selfdual and antiselfdual forms are obviously in direct sum; so $\mathcal{H}^2 = \mathcal{H}^2_+ \oplus \mathcal{H}^2_-$ (this decomposition corresponds to the ±1 eigenspaces of $* : \mathcal{H}^2 \to \mathcal{H}^2$).

If α is a nontrivial selfdual form then $\int_M \alpha\wedge\alpha = \int_M \alpha\wedge*\alpha = \int_M \langle\alpha,\alpha\rangle\, dvol_g = \|\alpha\|^2_{L^2} > 0$; and if $\beta \ne 0$ is antiselfdual then $\int_M \beta\wedge\beta = -\int_M \beta\wedge*\beta = -\|\beta\|^2_{L^2} < 0$. Moreover $\langle\alpha,\beta\rangle = \alpha\wedge*\beta = -\alpha\wedge\beta = -\beta\wedge\alpha = -\beta\wedge*\alpha = -\langle\beta,\alpha\rangle$, so $\alpha\wedge\beta$ is pointwise 0, and $\int_M \alpha\wedge\beta = 0$. Thus $\mathcal{H}^2_\pm$ are orthogonal and definite positive (resp. definite negative) for the intersection pairing.

b) At any point of M, the tangent space and the compatible triple (ω, J, g) can be identified with $(\mathbb{R}^4, \omega_0, J_0, g_0)$, with standard basis (e_1, e_2, e_3, e_4), and $J_0(e_1) = e_2$, $J_0(e_3) = e_4$. In terms of the dual basis,

$$\Lambda^2_+ = \mathrm{span}(e^1 \wedge e^2 + e^3 \wedge e^4,\ e^1 \wedge e^3 - e^2 \wedge e^4,\ e^1 \wedge e^4 + e^2 \wedge e^3),$$
$$\Lambda^2_- = \mathrm{span}(e^1 \wedge e^2 - e^3 \wedge e^4,\ e^1 \wedge e^3 + e^2 \wedge e^4,\ e^1 \wedge e^4 - e^2 \wedge e^3).$$

Meanwhile, $\omega = e^1 \wedge e^2 + e^3 \wedge e^4$, and $\Lambda^{2,0}$ is spanned by

$$(e^1 + ie^2) \wedge (e^3 + ie^4) = (e^1 \wedge e^3 \quad e^2 \wedge e^4) + i(e^1 \wedge e^4 + e^2 \wedge e^3),$$

while $\Lambda^{0,2}$ is the complex conjugate; it follows that $\Lambda^2_+ \otimes \mathbb{C} = \Lambda^{2,0} \oplus \Lambda^{0,2} \oplus \mathbb{C}\omega$. Moreover, the summands in this decomposition are clearly orthogonal (both for the standard Hermitian product $\langle \alpha, \beta \rangle = \alpha \wedge *\beta$ and for the complexified intersection pairing $(\alpha, \beta) \mapsto \alpha \wedge \beta$; in fact the two coincide in the selfdual case), as follows from considering the types.

Next, we observe that Λ^2_- is the orthogonal to Λ^2_+ (for either one of the two above-mentioned inner products on Λ^2); so $\Lambda^2_- \otimes \mathbb{C} = (\Lambda^{2,0} \oplus \Lambda^{0,2})^\perp \cap \omega^\perp = \Lambda^{1,1} \cap \omega^\perp$.

Let $\alpha \in \mathcal{H}^{1,1}_{\mathbb{R}}$ be a real harmonic $(1,1)$-form. Then $*\alpha$ is also a harmonic $(1,1)$-form, and hence so are α^+ and α^-. At every point of M we have $\Lambda^2_+ \cap \Lambda^{1,1} = \mathrm{span}(\omega)$, so $\alpha^+ = f\omega$ for some function $f : M \to \mathbb{R}$. Moreover, $d\alpha^+ = df \wedge \omega = 0$. However, exterior product with ω induces an isomorphism from Λ^1 to Λ^3, so $df \wedge \omega = 0$ if and only if $df = 0$. Therefore f is constant, and α^+ is a constant multiple of ω. We conclude that $\mathcal{H}^{1,1}_{\mathbb{R}} \subset \mathcal{H}^2 \oplus \mathbb{R}\omega$. Conversely, ω is a real (1,1)-form, and so is any antiselfdual form since $\Lambda^2_- \subset \Lambda^{1,1}$, so $\mathcal{H}^{1,1}_{\mathbb{R}} = \mathcal{H}^2_- \oplus \mathbb{R}\omega$.